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A study of muons deep underground
II. The rate of energy loss

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Abstract. The usual method of determining the rate of energy loss of muons by comparison of the sea-level spectrum with the intensity underground is shown to be inadequate to determine the energy-dependent term because of the imprecision of the spectrograph results. An alternative method has now been developed in which the rate of energy loss is deduced from the observed mean pulse height of a scintillation counter operated underground. This method requires a knowledge of the intensity–depth curve but not of the muon momentum spectrum. The various corrections necessary to apply the method to the results of the Tilmanstone Colliery experiment, described in the preceding paper, are considered and shown to be small. The energy loss by direct pair production is found to be 1·05 ± 0·20 times the value predicted by Kobayakawa.

1. Introduction

About twenty years ago it became evident that most cosmic-ray particles penetrating underground were muons and that, by comparing the momentum spectrum at sea level with the intensity of muons as a function of depth, it should be possible to determine the rate of energy loss of very high energy muons; the latter is usually assumed to be of the form

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = a(E)+b(E)E \tag{1}$$

where $a(E)$ is for the ionization loss and $b(E) = b_{\mathrm{b}}(E)+b_{\mathrm{p}}(E)+b_{\mathrm{n}}(E)$ for the bremsstrahlung, pair production and nuclear losses respectively. This early work is summarized in the review papers of Barrett *et al.* (1952) and George (1952). Since that time theoretical investigations have helped to substantiate the validity of the assumed forms of two of the terms contributing to the energy-loss equation—those for ionization and bremsstrahlung—but there is still no agreement on the magnitude of the pair production and nuclear terms. On the experimental side a number of further underground experiments have not greatly changed the picture at moderate depths, while the Kolar experiments have provided data at previously inaccessible depths. At the same time the momentum spectrum at sea level has been studied with increasingly sophisticated instruments. It should therefore have been possible by now to make a critical comparison of the momentum spectrum and the intensity–depth curve in order to establish the energy-loss expression experimentally.

In fact, this comparison can still only be carried out usefully for muons of energy up to about 100 GeV because the data from momentum spectrographs are still very limited at higher energies. There are two reasons for this, as can be seen in the most recent results from the Durham spectrograph (Aurela and Wolfendale 1967). Firstly, the aperture of this instrument was only 13 cm^2 sr, so that the number of traversals by particles of momentum greater than 300 GeV/c was only about 100 per year. Secondly, since the maximum *detectable* momentum was about 1000 GeV/c, the uncertainty in the classification of particles above 100 GeV/c must be very large (see Bull *et al.* 1965, Stefanski *et al.* 1968). Consequently Aurela and Wolfendale only used their data for the spectrum at momenta below 100 GeV/c and extrapolated it to high momenta by using the intensity–depth data and an *assumed* form of the energy-loss expression. For energies less than 100 GeV the ionization loss is much larger than any of the other three terms, so that the existing momentum spectrographs have provided only broad upper limits for their magnitudes.

It is therefore essential to study these energy-loss processes by some other experimental method. At low energies the cross sections can be measured directly in accelerator or cosmic-ray experiments, but their extrapolations to higher energies are very uncertain. A few years ago we suggested (Barton and Stockel 1966) that an alternative approach was to study the mean energy dissipated in a scintillation counter operated underground; it was shown that this was proportional to the product of the b term in the energy-loss expression and the mean muon energy at the point of observation. This approach was later criticized (Rochester, private communication) as the mean muon energy could itself only be estimated by assuming the energy-loss expression. The theoretical section of the present paper will show that a modified theory can circumvent this difficulty. The following section considers the application of the theory and the corrections which need to be made before comparing it with the data obtained from the experiment described in the preceding paper (Stockel 1969).

In the earlier work it was customary to assume that $b(E)$ in equation (1) was independent of energy, but it is now known theoretically that this is a rather drastic approximation. Before developing the theory of the new method it is therefore convenient to review briefly the present theoretical estimates for the various terms in the energy-loss expression.

2. Theoretical energy-loss expression

The various terms in this expression have been reviewed by many authors. Recently a detailed account has been given by Kobayakawa (1967), and this forms the basis for the present work. All values refer to 'standard rock' with $Z = 11$, $A = 22$ and $\rho = 2{\cdot}65$.

2.1. *Ionization and excitation*

It is customary to quote Sternheimer (1956) as justification for writing

$$a(E) = 1{\cdot}888 + 0{\cdot}0768 \ln\left(\frac{E_{\mathrm{m}}'}{\mu}\right) \text{ MeV g}^{-1}\,\text{cm}^2 \tag{2}$$

where E_{m}' is the maximum transferable energy in GeV and μ the muon rest mass energy.

This only applies above 100 GeV and it is therefore preferable to use the complete Sternheimer formula

$$a(E) = \frac{A'}{\beta^2}\left[B + C + 0{\cdot}69 + \ln(1000E_{\mathrm{m}}') - 2\beta^2 - a_0\left\{\lg\left(\frac{E}{\mu}\right) - 3\right\}^m\right] \tag{3}$$

where $\beta = v/c$ and the last term is included only if $E < 100$ GeV. A', B, C, a_0 and m are constants given by Sternheimer for a wide range of materials. A' is related directly to Z/A for the material, but B and C depend on the mean ionization potential of the atoms concerned and are therefore not uniquely determined for 'standard rock'. It seems best to take the values of B and C for aluminium as it has approximately the correct density. By examining Sternheimer's values for other materials and considering the composition of various rocks, it is estimated that the possible resulting error should not be more than 1%.

2.2. *Bremsstrahlung*

Kobayakawa finds

$$\begin{aligned} b_{\mathrm{b}}(E) &= 1{\cdot}795 \times 10^{-7}\{\ln(E/\mu) - 0{\cdot}257\}\ \text{g}^{-1}\,\text{cm}^2 \text{ for } E < 2460 \text{ GeV} \\ &= 1{\cdot}759 \times 10^{-6}\ \text{g}^{-1}\,\text{cm}^2 \text{ for } E \geqslant 2460 \text{ GeV}. \end{aligned} \tag{4}$$

These values are rather similar to those calculated by others. For example, interpolating between values given by Erlykin (1966) for $Z = 10$ and 13 leads to estimates about 5% higher. However, Petrukhin and Shestakov (1968) have suggested that the effects of screening have not so far been taken into account correctly and that all the values need to be reduced by 20%.

2.3. *Pair production*

Using a theory of Murota *et al.* (1956) Kobayakawa finds

$$b_p(E) = 4{\cdot}154\times10^{-7}\left\{\ln\left(\frac{E}{\mu}\right)-2{\cdot}787\right\}\ \mathrm{g^{-1}\,cm^2}\ \text{for } E < 25{\cdot}5\ \text{GeV}$$

$$= 2{\cdot}066\times10^{-6}\left\{\frac{\ln(E/\mu)-2{\cdot}787}{\ln(E/\mu)-0{\cdot}485}\right\}\ \mathrm{g^{-1}\,cm^2}\ \text{for } E \geqslant 25{\cdot}5\ \text{GeV}. \qquad (5)$$

There seems considerably greater uncertainty about this result as the Murota *et al.* analysis does not lead to readily computed values. Erlykin (1966), basing his calculations on the theory of Ternovski (1959), deduced estimates which are about 40% higher for E between 100 and 10 000 GeV, while Kelner and Kotov (1968) found even higher values.

2.4. *Nuclear interactions*

Kobayakawa has computed values which can be fitted by the expression

$$b_n(E) = (0{\cdot}243+0{\cdot}007\ln E)\times10^{-6}\ \mathrm{g^{-1}\,cm^2} \qquad (6)$$

where E is in GeV. It must be stressed that there is still great uncertainty about this estimate. Firstly, although the theory of Daiyasu *et al.* (1962), on which it is based, is probably an advance over earlier theories, the choices for the proportion of the 'core-like' interaction and for the effective radius of the 'cloud-like' interaction are to a large extent arbitrary. Secondly, it is assumed that the photonuclear cross section is constant at 72 μbn, whereas the experimentally measured values (Hilpert *et al.* 1968) for energies below 5 GeV are considerably higher than this. The cross section is certainly falling with increasing energy in the region of a few GeV but there is no direct experimental evidence bearing on its behaviour at high energies. Murdoch and Rathgeber (1964) have suggested that it may fall inversely with energy, and there does not seem to be any strong theoretical argument for ruling out this possibility. It is concluded that Kobayakawa's estimate is probably the best that can be made at the present time, but that it might well be in error by 50% in either direction.

3. Theory of mean energy dissipation

At the depth of observation h we first calculate the mean energy dissipated in a thin layer δh. This is given by

$$\delta E_1 = \frac{\int_{E=0}^{\infty} N(E,h)(\mathrm{d}E/\mathrm{d}x)\delta h\,\mathrm{d}E}{\int_{E=0}^{\infty} N(E,h)\,\mathrm{d}E} \qquad (7)$$

where $N(E, h)$ is the differential energy spectrum of muons at the depth h and $\mathrm{d}E/\mathrm{d}x$ is the sum of the four terms given in § 2.

Carrying out the integration by parts of the numerator, this gives

$$\frac{\delta E_1}{\delta h} = \frac{-[I(E,h)\,\mathrm{d}E/\mathrm{d}x]_0^{\infty} + \int_0^{\infty} I(E,h)(\mathrm{d}/\mathrm{d}E)(\mathrm{d}E/\mathrm{d}x)\,\mathrm{d}E}{-[I(E,h)]_0^{\infty}} \qquad (8)$$

with $I(E, h)$ as the integral spectrum of muons at the depth h. Therefore

$$\frac{\delta E_1}{\delta h} = \left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{E=0} + \frac{1}{I(0,h)}\int_0^{\infty} I(E,h)\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)\mathrm{d}E. \qquad (9)$$

(If $\mathrm{d}E/\mathrm{d}x$ is simplified to the form $a+bE$ it is clear that it is the second term which depends on b.) The difficulty in evaluating the integral is that even at sea level the spectrum of muons is not well established at energies above 100 GeV. However, the integral energy spectrum can always be replaced by an integral range spectrum if the range–energy relationship is known. Of course, this requires values for $\mathrm{d}E/\mathrm{d}x$, which is the quantity that has to be determined, but the argument is not a circular one as we can then calculate the mean energy dissipated as a function of the assumed expression for $\mathrm{d}E/\mathrm{d}x$. It will be shown in § 6 that the numerical values obtained are quite sensitive to the uncertain parameters in this expression.

So, in computing the integral, $I(E, h)$ is replaced by $I(0, h'+h)$, where

$$h' = \int_0^E \frac{\mathrm{d}E}{a(E)+b(E)E} \tag{10}$$

and the values for the intensity are taken from the expression given in the previous paper:

$$I = A_0 \frac{h^{-\alpha}}{h+h_1} \mathrm{e}^{-\beta h} \tag{11}$$

with $A_0 = 45$, $\alpha = 1{\cdot}38$, $h_1 = 162{\cdot}5$, $\beta = 0{\cdot}000\,705$. The values used for $a(E)$, $b(E)$ and $(\mathrm{d}/\mathrm{d}E)(\mathrm{d}E/\mathrm{d}x)$ are deduced from the expressions given in § 2.

The calculation so far has assumed that all the particles are travelling vertically. It can easily be shown that a $\cos^n\theta$ distribution of particles traversing a horizontal layer of thickness δh has a mean track length of $\delta h\{1+1/(n+1)\}$, which increases the energy dissipation by the same factor. There is a further increase because the inclined radiation is of relatively higher energy. Both effects are allowed for by replacing equation (9) by

$$\frac{\delta E_1}{\delta h} = \frac{\displaystyle\int_0^{\pi/2} \sec\theta \left\{ \left.\frac{\mathrm{d}E}{\mathrm{d}x}\right|_{E=0} + \frac{1}{I(0, h\sec\theta)} \int_0^\infty I(E, h\sec\theta) \frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\mathrm{d}E}{\mathrm{d}x}\right) \mathrm{d}E \right\} 2\pi \sin\theta \cos^{n+1}\theta \,\mathrm{d}\theta}{\displaystyle\int_0^{\pi/2} 2\pi \sin\theta \cos^{n+1}\theta \,\mathrm{d}\theta}. \tag{12}$$

Finally, a correction must be made for the effect of fluctuations on the range–energy relationship. This problem has been studied by many authors, and the most recent Monte Carlo studies of Osborne *et al.* (1968) agree with Kobayakawa's analytical results. The consequence of the energy-loss fluctuations is that the range spectrum leads to an overestimate of the integral energy spectrum and hence of the estimated mean energy dissipation. However, the major part of the contribution to the integral in the above expression arises from particles which stop within 1000 hg cm^{-2} of the point of observation, whereas the effect of fluctuations is only large for muons penetrating much further. Using Kobayakawa's values, it was found that the correction was negligible at shallow depths and only about 1% at 2235 hg cm^{-2}.

4. Experimental results

The pulse-height histograms, similar to that given in § 3 of the preceding paper, were converted to a linear scale using the known incremental channel width of the analyser and taking channel number 20 to have unit width. The mode pulse height for each run was then obtained by carefully fitting a smooth curve to a large-scale diagram of the distribution for single events and measuring the position of the maximum ordinate. This graphical method was adopted for determining the mode, as it has been shown by Barnaby (1961) to give reliable results for this type of counter. The results obtained are shown in table 1. It is not possible to compare the mode values obtained at the various depths as the temperatures were considerably different and this affected the threshold of the pulse-height analyser.

Table 1. Energy-loss values

Depth ($hg\ cm^{-2}$)	Mode pulse height (linear scale units)	Mean pulse height (linear scale units)	mean/mode
71·6	0·84 ± 0·01	1·29 ± 0·02	1·54 ± 0·04
734	0·67 ± 0·01	1·19 ± 0·03	1·78 ± 0·06
1068	0·66 ± 0·01	1·15 ± 0·02	1·75 ± 0·04
2235 (floor)	0·76 ± 0·01	1·36 ± 0·03	1·79 ± 0·06
2235 (roof)	0·76 ± 0·01	1·44 ± 0·03	1·90 ± 0·06

The mean energy loss was obtained by numerical integration of the area under the energy-loss spectrum for the total events in each run. For the purpose of the analysis given in this paper an upper cut-off was applied at a channel corresponding to the detection of a shower of 35 particles. The ratio of mean to mode energy loss was then taken because there was no direct method of converting the pulse heights to energy units.

5. Comparison of theory with experiment

The theory of § 3 estimates the mean energy dissipated by muons in a thin horizontal layer of rock at any depth underground. Experimentally, the information available is in the form of a pulse-height distribution from a scintillator. Normalizing the area under this distribution leads directly to a mean energy, but there are clearly a number of points which must be considered before comparing this mean energy with that computed theoretically. The discussion will be carried out with reference to the particular apparatus described in the preceding paper.

(i) An ideal apparatus would use a scintillator of composition and density similar to those of the rock. The plastic scintillator used does not satisfy this requirement, but any transition effect should be very small. An upper limit for the correction can be made by calculating the ratio of mean to mode energy loss for ionization loss only in both standard rock and the plastic scintillator. This ratio is 2% higher in the scintillator, which indicates that the effect is negligible for thicknesses much less than a radiation length.

(ii) The counter is only triggered by particles passing through the top and bottom Geiger counter trays. This has two effects. Firstly, it restricts the aperture of the telescope, which can be taken into account when evaluating the integral over angle in equation (12). Secondly, it means that solitary electrons of energy less than about 7 MeV fail to be recorded and included in the pulse-height distribution. This would have less importance for a large area counter, as the muon giving rise to a low-energy electron would be more likely to trigger the apparatus and a greater fraction of the total energy would be recorded, but for a small counter the parent muon might pass outside it. Even for an apparatus of the size used in this experiment (90 cm × 40 cm) this effect was quite small. The actual number was estimated by examining the hodoscope records at each depth for events in which at least one additional single Geiger counter was discharged. In this way some idea of the 'structure function' of low-energy electrons and photons could be established and the necessary corrections thus determined. The correction was 3% for the 'floor' experiment at 2235 hg cm^{-2} and less in all other cases.

(iii) The theory includes the energy dissipation by all muon interactions, but the apparatus has only a limited dynamic range, so that a very large secondary shower cannot be recorded correctly. In practice the number of such interactions is very small, but their contribution to the energy loss is not. Indeed, the contribution to the mean energy of a few very large showers is so important that, at least for the present results, it is necessary to impose a maximum energy loss which is less than that enforced by saturation of the photomultipliers or analyser; otherwise the uncertainty in the assessment of the mean would be large as the statistics of very great energy secondaries is inevitably poor.

The mean energies quoted in the previous section were therefore calculated with a cut-off equivalent to 35 minimum ionizing particles traversing the counter. Shower cascade curves for rock are not available, but interpolating between the Monte Carlo results of Butcher and Messel (1960) for air and aluminium it is estimated that a shower of 20 GeV has about 35 particles at its maximum development. This means that the counter records all interactions giving rise to secondaries of less than 20 GeV, but higher-energy showers are only included if they are observed early or late in their development. From the form of the shower curves it is then deduced that higher-energy vertical showers only contribute 20 GeV to the energy integral. This correction to the theoretical estimates of the mean energy is unimportant at shallow depths, but removes most of the bremsstrahlung contribution at greater depths where the median muon energy is much larger than 20 GeV. The reduction in the knock-on contribution is quite small, because the probability of a secondary energy E' falls as $1/E'^2$ instead of $1/E'$, but is readily calculated.

For direct pair production there is no certainty about the dependence on E', but the theories of Bhabha (1935) and Murota *et al.* (1956) agree in predicting that for $E' > 0{\cdot}01E$ there is a rapid decrease in cross section as $1/E'^3$. Since the proportion of muons of energy greater than 2000 GeV is never more than 0·5%, it is believed that substantially all of this contribution is included. The contribution from nuclear interactions is less certain. The dependence on E' is similar to that for bremsstrahlung, but in the first collision on average only one-third of the energy goes into the production of neutral pions and thence the electromagnetic component. The charged pions will interact further down (the probability of decay in the rock is small) and produce more showers, which will be superimposed on the original ones as the pion interaction length is about double the radiation length in rock. In the cascade of pions and nucleons some energy is dissipated by heavily ionizing particles, for which plastic scintillators give a relatively smaller light output. It does not seem possible to estimate these effects reliably, but, as they are both small and affect the nuclear contribution in opposite directions, they are less significant than the differences between the various theories of the muon–nuclear interaction.

(iv) Since it is difficult to calibrate a plastic scintillator directly for the energy dissipated, the quantity measured experimentally in this experiment was the ratio of mean to mode pulse heights. The Sternheimer theory can be used, together with an appropriate muon energy spectrum for each case, to predict the mode energy for particles travelling vertically. The experimental values will depend also on the angular distribution, and a correction must be made for this. Although both mean and mode are altered in the same direction when allowing for the angular distribution, the former is affected more strongly, so that it is not sufficient merely to predict the ratio for vertical particles.

The corrections detailed above have been applied to the theoretical estimates, except for (ii), which it was more convenient to apply to the experimental data so that the two readings at 2235 hg cm^{-2} could be combined into a single one with reduced error.

6. Results and conclusions

Using the data from § 4 and the theoretical values calculated as described above, the corrected ratios are given below.

Depth (hg cm^{-2})		71·6	734	1068	2235
mean/mode	Experiment	1·55 ± 0·04	1·81 ± 0·06	1·79 ± 0·04	1·89 ± 0·04
	Theory	1·52	1·75	1·80	1·88

The agreement is seen to be satisfactory, but it is necessary to discuss how sensitive the agreement is to uncertainties in the theory. In principle the theory of § 3 is an exact one, but uncertainties in the corrections introduced in the subsequent section mean that the theoretical estimates have an estimated precision of only 3%, comparable with the error on the experimental results. More important are possible changes in the assumed forms for the various energy-loss processes. It is unlikely that the ionization-loss formula is incorrect by more than 1% (Crispin and Fowler 1969), and there is general agreement about the bremsstrahlung formula. The effect of varying the nuclear or pair production losses is shown in figure 1. The assumption made here is that the functional form of these losses remains the same, but that their absolute magnitudes are larger than assumed by Kobayakawa. Clearly

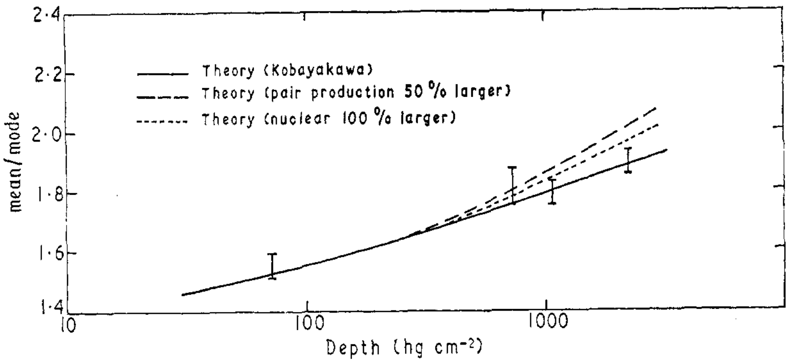

Figure 1. Variation of ratio of mean loss to mode loss with depth.

the experiment is not very sensitive to the nuclear-loss process, so that even doubling its value is not excluded. On the other hand, the mean energy loss does depend strongly on the pair production losses, and from the table above it is unlikely that they could be much larger than Kobayakawa predicts. If his estimates of the nuclear contribution are too low (they are considerably less than the Kessler and Kessler (1956) values), the pair production one might even have to be reduced. This is interesting as his values for the pair production contribution to b, $1{\cdot}3\times10^{-6}$ at 100 GeV and $1{\cdot}5\times10^{-6}$ at 1000 GeV, are already less than in many earlier works. If we assume that the nuclear contribution has been correctly assessed, our result can be stated in the form

$$\frac{\text{experimental pair production loss by muons}}{\text{theoretical pair production loss by muons}} = 1{\cdot}05\pm0{\cdot}20.$$

This result clearly favours the Murota *et al.* (1956) theory rather than that of Erlykin (1966) or Kelner and Kotov (1968), which predict losses from 40% to 60% larger respectively. It follows that Kobayakawa's results for the range–energy relation for muons can be regarded as more reliable than earlier estimates, and that the sea-level spectrum which he has recently derived (Kobayakawa 1968) from the intensity–depth results must be substantially correct.

Bergeson *et al.* (1968) have proposed that the integral muon spectrum needs to be raised considerably at energies greater than 10^{12} eV; their value for the muon intensity is double Kobayakawa's at 3×10^{12} eV. The difference arises from their assumption that the loss from nuclear interactions is about five times larger at these energies than has been assumed here. This is quite incompatible with the present experiment unless the increase in cross section sets in rather sharply above 500 GeV.

In conclusion, it is clear that the method used in this paper could be developed further. The present apparatus was not designed with this method in mind, but has already provided results of useful accuracy. It should therefore be possible to make a more precise measurement using a larger counter and one in which the calibration is obtained directly rather than from the mode energy loss.

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References

Aurela, A. M., and Wolfendale, A. W., 1967, *Ann. Acad. Scient. Fenn.* A, VI, **227**, 1–14.
Barnaby, C. F., 1961, *Proc. Phys. Soc.*, **77**, 1149–56.
Barrett, P. H., *et al.*, 1952, *Rev. Mod. Phys.*, **24**, 133–78.

Barton, J. C., and Stockel, C. T., 1966, *Proc. 9th Int. Conf. Cosmic Rays, London*, 1965 (London: Institute of Physics and Physical Society), pp. 992–6.
Bergeson, H. E., *et al.*, 1968, *Phys. Rev. Lett.*, **21**, 1089–93.
Bhabha, H. J., 1935, *Proc. R. Soc.* A, **152**, 559–86.
Bull, R. M., Nash, W. F., and Rastin, B. C., 1965, *Nuovo Cim.*, **40**, 365–84.
Butcher, J. C., and Messel, H., 1960, *Nucl. Phys.*, **20**, 15–128.
Crispin, A., and Fowler, G. N., 1969, *Rev. Mod. Phys.*, submitted for publication.
Daiyasu, K., Kobayakawa, K., Murota, T., and Nakano, T., 1962, *J. Phys. Soc. Japan* (*Suppl.* A-III), **17**, 344–7, 367–8.
Erlykin, A. D., 1966, *Proc. 9th Int. Conf. Cosmic Rays, London*, 1965 (London: Institute of Physics and Physical Society), pp. 999–1002.
George, E. P., 1952, *Prog. Cosmic Ray Phys.*, **1**, 393–451 (Amsterdam: North-Holland).
Hilpert, H. G., *et al.*, 1968, *Phys. Lett.*, **27B**, 474–6.
Kelner, S. R., and Kotov, Yu. D., 1968, *Can. J. Phys.*, **46**, S387–90.
Kessler, D., and Kessler, P., 1956, *C.R. Acad. Sci., Paris*, **242**, 3045–7.
Kobayakawa, K., 1967, *Nuovo Cim.*, **47**, 156–84.
—— 1968, *Can. J. Phys.*, **46**, S395–8.
Murdoch, H. S., and Rathgeber, H. D., 1964, *Phys. Lett.*, **13**, 267–8.
Murota, T., Ueda, A., and Tanaka, H., 1956, *Prog. Theor. Phys., Japan*, **16**, 482–96.
Osborne, J. L., Wolfendale, A. W., and Young, E. C. M., 1968, *J. Phys.* A. (*Proc. Phys. Soc.*), [2], **1**, 55–60.
Petrukhin, A. A., and Shestakov, V. V., 1968, *Can. J. Phys.*, **46**, S377–80.
Stefanski, R. J., Adair, R. K., and Kasha, A., 1968, *Phys. Rev. Lett.*, **20**, 950–2.
Sternheimer, R. M., 1956, *Phys. Rev.*, **103**, 511–5.
Stockel, C. T., 1969, *J. Phys.* A (*Gen. Phys.*), [2], **2**, 639–49.
Ternovski, F. F., 1959, *Zh. Eksp. Teor. Fiz.*, **37**, 793–8 (*Sov. Phys.-JETP*, **10**, 565–9).